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HYDRODYNAMICS OF A STRATIFIED LIQUID IN THE TERMINOLOGY OF THE LAMB MOMENTUM DENSITY

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The wave motion of a stratified fluid is not separated from the vortex component in the Navier—Stokes equations. This makes the analysis of motion difficult in the nonlinear case when the wave and vortex components can reciprocally generate each other. Consequently, a description of the nonlinear dynamics of a stratified fluid in the terminology of the velocity or vorticity fields is not optimal and selection of other variables, whose evolution in time would be mutually less dependent, is desirable.

As is shown in [1, 2], a particular class of ideal stratified media motions exists which conserve their form under arbitrary levels of nonlinearity. In an incompressible fluid these are the motions whose velocity field can be expressed in terms of the density ρ and scalar functions λ, φ by the formula [2]

$$\rho \mathbf{v} = -\nabla\varphi + \lambda\nabla\rho. \tag{1}$$

In the terminology of the functions introduced, the fluid dynamics turns out to be Hamiltonian while λ, ρ are canonically conjugate variables. The wave motions that are described by such variables possess vorticity. However, the class of motions (1) is constrained, and they can be considered analogs of potential motions of a homogeneous fluid [2].

In this paper, a representation is obtained for the velocity field of an incompressible fluid, which generalizes (1) and yields a partition of the total motion into separate components. This representation results in a natural manner from the equations of motion if they are first written in the terminology of a new variable, the Lamb momentum density. The equations obtained are converted to Hamiltonian form. They can be used to search for the Lagrange

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and integral invariants by using a procedure analogous to that proposed in [3, 4] for the homogeneous fluid case.

LAMB MOMENTUM DENSITY OF AN INHOMOGENEOUS FLUID

The dynamics of an ideal inhomogeneous fluid is described by the Euler equations

$$d(\rho\mathbf{v})/dt = -\nabla p - \rho\mathbf{g}; \tag{2}$$

$$\operatorname{div}\mathbf{v} = 0,\ d\rho/dt = 0, \tag{3}$$

where $d/dt = \partial/\partial t + (\mathbf{v}\nabla)$, and g is the acceleration of the gravity force in the field. The remaining notation is standard. We make a change of variables in (2) and (3). Instead of the velocity $\mathbf{v}$ and pressure p we introduce the fields q, φ by means of the formula

$$\rho\mathbf{v} = -\nabla\varphi + \mathbf{q}. \tag{4}$$

The vector relation (4) must be supplemented by a scalar relationship which is selected in such a manner that the pressure gradient is eliminated from the equation of motion (see (8) below). If the fields q, ρ are given, then by using the incompressibility condition

$$\operatorname{div}\mathbf{v} = \operatorname{div}\left[(-\nabla\varphi + \mathbf{q})/\rho\right] = 0 \tag{5}$$

the variable φ can be found, and when known $\mathbf{v}$ can be found from (4). Therefore, the dynamics of the fluid is described completely by the fields q, ρ. The field $\mathbf{q}$ plays the part of the Lamb momentum density.

As is known [5, 6], the fluid motion can be characterized by the magnitude of the Lamb momentum, the total momentum of the force which is required to generate motion from the state of rest. Let us examine the motion that occurs in an inhomogeneous fluid under the action of a pulse external force

(6)

where $\delta(t)$ is the Dirac function, and $\mathbf{q}(\mathbf{x})$ is the momentum density which is transmitted instantly to the fluid. Its modulus is considered a bounded function of the coordinates. Let us add the force determined by (6) into the right side of (2) and let us integrate over a small time interval $(-\varepsilon, \varepsilon)$. Since the velocities of the motions that occur are finite, in the limit $\varepsilon \to 0$ the convective term in (2) and the term with the gravity force field yield no contribution. In general, the pressure in an incompressible fluid subjected to the action of a pulse load will contain a contribution proportional to $\delta(t)$. Consequently, the result of the integration is written in the form (4), where $\varphi = \lim\limits_{\varepsilon\to 0}\int p(t)\,dt$. The ordinary momentum density is a particular case of the Lamb momentum density: $\mathbf{q}(\mathbf{x})$ can always be selected in such a way that $\rho\mathbf{v} = \mathbf{q}$. In this case $\varphi = 0$. In contrast to q the velocity $\mathbf{v}$ should satisfy the additional condition of being solenoidal. Consequently, this field $\mathbf{v}$ can be generated by many $\mathbf{q}$ distributions. All such distributions are distinguished by the gradient of the scalar function that is determined by the scalar calibration condition and is selected from considerations of convenience.

Let us deduce the dynamical equation for $\mathbf{q}$. Substitution of (4) into (2) yields

$$\frac{dq_i}{dt} = -q_j\frac{\partial v_j}{\partial x_i} + \frac{\partial}{\partial x_i}\left[-p - \rho\mathbf{g}\mathbf{x} + \frac{d\varphi}{dt} + \frac{1}{2}\rho v^2 - \Pi(\rho)\right] + \left(\mathbf{g}\mathbf{x} + \frac{d\Pi}{d\rho} - \frac{1}{2}v^2\right)\frac{\partial\rho}{\partial x_i}. \tag{7}$$

Analogously to [2], the density function $\Pi(\rho)$ is introduced here, which is still considered arbitrary. As the calibration condition we select an equation analogous to the Cauchy–Lagrange integral

$$d\varphi/dt = p + \rho\mathbf{g}\mathbf{x} - \rho v^2/2 + \Pi(\rho). \tag{8}$$

Equation (7) acquires the form

$$dq_i/dt = -q_j\partial v_j/\partial x_i + (\mathbf{g}\mathbf{x} + d\Pi/d\rho - v^2/2)\partial\rho/\partial x_i, \tag{9}$$

which completely governs the dynamics of an ideal inhomogeneous fluid. The fields φ, $\mathbf{v}$ are expressed in terms of $\mathbf{q}$ from (4) and (5). The pressure is found from the calibration condition (8).

Let us examine certain properties of the field $\mathbf{q}$ and the equations obtained in the simplest case of a homogeneous fluid ρ = const. In this case, the nonessential constant Π and the term with the gravity force field can be neglected in (8), while the term with the gradient of ρ drops out in (9):

$$dq_i/dt = -q_j\partial v_j/\partial x_i. \tag{10}$$

It follows from (4) that in the domain where $\mathbf{q} = 0$ the flow is potentially $\mathbf{v} = -\nabla\varphi/\rho$. In general, the converse is not true. Let the potential motion domain being considered be simply connected (slits are assumed drawn in the case of a multiconnected domain). Substituting $\mathbf{v} = -\nabla\Phi/\rho$ into the Euler equation, we obtain the Cauchy–Lagrange integral $\partial\Phi/\partial t - \rho v^2/2 - p = \text{const}$, or

$$d\Phi/dt = p - \rho v^2/2 + \text{const}. \tag{11}$$

If the nonessential constant is omitted in the right side of (11), then $(d/dt)(\Phi - \varphi) = 0$. It is easy to confirm that if $\mathbf{q} = \mathbf{q}_1$ is a solution of (10), then $\mathbf{q}_1 + \nabla I$, where I will also be a solution of (10), where I is an arbitrary function satisfying $dI/dt = 0$. If we select $I = \Phi - \varphi$, then the new field turns out to equal zero outside the slits and the flow domains with vorticity. It hence follows that a finite vorticity distribution can be substituted in conformity with the finite distribution $\mathbf{q}(\mathbf{x})$.

According to (4), in the case ρ = const the field $\mathbf{q}$ is the sum of the solenoidal $\rho\mathbf{v}$ and gradient $\nabla\varphi$ components. The solenoidal and gradient components of any vector field can be extracted by using projection operators. For instance, their explicit form is known in the case of an unbounded domain with fields that decrease sufficiently rapidly at infinity [6]:

$$\partial\varphi/\partial x_i = \int \Pi_{ij}(\mathbf{x}, \mathbf{x}')\, q_j(\mathbf{x}')\, dV(\mathbf{x}'),$$

$$v_i(\mathbf{x}) = \int Q_{ij}(\mathbf{x}, \mathbf{x}')\, q_j(\mathbf{x}')\, dV(\mathbf{x}'), \tag{12}$$

$$\Pi_{ij} = -\frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{|\mathbf{x} - \mathbf{x}'|}, \quad Q_{ij} = \delta_{ij}\delta(\mathbf{x} - \mathbf{x}') - \Pi_{ij}.$$

In a bounded domain, the Green's function of the boundary value problem for the Poisson equation should be used in place of $1/|\mathbf{x} - \mathbf{x}'|$.

Now, let the fluid be inhomogeneous, $\rho \neq$ const. We consider the evolution of the field $\zeta = \mathbf{q} \times \nabla\rho$. A direct calculation utilizing (9) $d\rho/dt = 0$ shows that ζ satisfies the equation

$$d\zeta/dt = (\zeta\nabla)\mathbf{v}, \tag{13}$$

which is analogous to the equation for the vorticity in a homogeneous fluid. It follows from the form of (13) that ζ belongs to the class of frozen fields whose force lines move together with the fluid. If $\zeta = 0$ at the initial time, then it equals zero also in all succeeding times. In this case $\mathbf{q}$ is parallel to $\nabla\rho$:

$$\mathbf{q} = \lambda\nabla\rho, \quad \rho\mathbf{v} = -\nabla\varphi + \lambda\nabla\rho, \tag{14}$$

where λ is a certain scalar function. Substitution of (14) into (9) yields a dynamic equation for λ

$$d\lambda/dt = -v^2/2 + d\Pi/d\rho + \mathbf{g}\cdot\mathbf{x}. \tag{15}$$

The representation (14) is in agreement with (1) and, in particular, describes an internal wave. The function $\Pi(\rho)$ can be determined from the condition that the hydrostatic equilibrium be described by trivial solutions of the motion equation [2].

In the general case, we set

$$\mathbf{q} = \lambda\nabla\rho + \mathbf{q}', \quad \rho\mathbf{v} = -\nabla\varphi + \lambda\nabla\rho + \mathbf{q}' \tag{16}$$

in place of (14), where λ satisfies (15). Substituting (16) into (9) we obtain

$$dq_i'/dt = -q_j'\partial v_j/\partial x_i, \tag{17}$$

that is, $\mathbf{q}'$ satisfies the same equation as $\mathbf{q}$ in a homogeneous fluid. The representation (16) yields a convenient partition of the total Lamb momentum density into two components. The first can be utilized to describe internal waves, and the other, all other fluid motions.

LAGRANGE INVARIANTS

Let us examine fields of three kinds. The first will be "Lagrange invariants," the scalar functions $I^{(l)}(\mathbf{x}, t)$, $l = 1, 2, \ldots$ that are invariant along Lagrange trajectories

$$dI/dt = 0. \tag{18}$$

Another kind is the vector "frozen fields" $\mathbf{J}^{(m)}(\mathbf{x}, t)$, m = 1, 2, ... that satisfy the equation

$$dJ_i/dt = J_j \partial v_i/\partial x_j. \tag{19}$$

Equation (19) shows that the field J evolves along Lagrange trajectories analogously to the vector dl, connecting two infinitely close material points [6]. The vorticity ω = rot **v** in a homogeneous fluid and the field ζ in an inhomogeneous fluid (see (13)) satisfy (19). Together with I and J, we consider the fields $\mathbf{S}^{(n)}(x, t)$, n = 1, 2, ... that evolve along Lagrange trajectories similarly to differential oriented area elements [6]:

$$dS_i/dt = -S_j \partial v_j/\partial x_i. \tag{20}$$

The field **q** for a homogeneous fluid and the field **q'** for an inhomogeneous fluid are an example of the fields S.

It is shown in [3] how new fields are constructed from known Lagrangian invariants and frozen fields. Thus, the field $I\nabla I' \times \nabla I''$ is a frozen field and the scalar function $(J\nabla)I$ is a new Lagrangian invariant. The Jacobian of the three fields $D(I^{(1)}, I^{(2)}, I^{(3)})/D(x_1, x_2, x_3)$ will also be a Lagrangian invariant. Successively applying these relationships and considering linear combinations of the fields, we can obtain new Lagrangian invariants and new frozen fields.

Equation (20) can be utilized to search for additional Lagrange invariants, and the scheme of their search becomes more symmetrical. By virtue of the quasilinearity of (18)-(20), linear combinations of the fields of each species belong to the same class. Direct calculations show that the fields I, J, S possess the following properties. The gradient of the function **I** satisfies (20). The curl of the field **S** is a frozen field, while the divergence of any frozen field is a Lagrange invariant. For instance, div $\zeta = \rho e$, where $e = \nabla\rho\cdot\omega$ is the Ertel Lagrange invariant [7].

It can also be shown that the fields J, S possess the following reciprocity properties. The vector product $\mathbf{S} \times \mathbf{S}'$ is a frozen field while the vector product of the two frozen fields $\mathbf{J} \times \mathbf{J}'$ satisfies (20). The scalar product $\mathbf{J}\cdot\mathbf{S}$ is a Lagrange invariant. Using all these relationships, new fields can be constructed from known fields of the type **I**, **J**, **S**.

Let us consider simple examples. The vector **dl** connecting two infinitely nearby material points satisfies (19) while the field **q'** satisfies (20). Hence, their scalar product **q**·**dl** is a Lagrange invariant. The integral of **q'** along an arbitrary material contour (not certainly closed) is also conserved in time. The field **q'**·curl **q'** is naturally called the spirality of the field **q'**. According to the exposition above, curl **q'** is a frozen field while **q'**·curl **q'** is a Lagrange invariant. Therefore, in contrast to the spirality of the velocity field **v**·curl **v**, which is conserved integrally [8], the spirality of the field **q'** is conserved locally, along the Lagrange trajectories.

At the initial time let **q'**·curl **q'** = 0. Then the spirality of the field **q'** equals zero even in subsequent times. As is known [8, 9], the spirality of the field equals zero in any case if this field is representable in the form. Putting $q' = \chi\nabla\psi$ in (16), we obtain the relationship

$$\rho\mathbf{v} = -\nabla\varphi + \lambda\nabla\rho + \chi\nabla\psi. \tag{21}$$

As is noted in [10], the equality (21) can be considered a generalization of the representation (1) and the Clebsch representation (in the particular case of $\nabla\rho = 0$, (21) agrees with the Clebsch representation [5, 11]). Any Lagrange invariant, the Ertel invariant e, say [12], can be selected as the potential ψ. Substituting $q' = \chi\nabla e$ into (17), we obtain that χ should also be a Lagrange invariant. The dynamics of the fluid is described in terms of five functions while the total system of equations consists of (5), (15) and three equations

$$d\rho/dt = de/dt = d\chi/dt = 0. \tag{22}$$

The boundedness of the Clebsch representation and its generalization (21) is clear from the exposition above: They describe just flows with spirality of the field **q'**, which equals zero everywhere. A more general class of flows can be described at the cost of introducing multivalued Clebsch potentials [13] while remaining within the framework of the representation (21).

HAMILTONIAN FORM OF THE MOTION EQUATIONS

The total energy of a stratified fluid is written in the form

$$H = \int [\rho v^2/2 + U(\rho, \mathbf{x})]\, dV, \tag{23}$$

where U is the potential energy density. Since the potential energy is determined to the accuracy of a constant, then an arbitrary function of the Lagrange invariants of motion can be included in U, for instance, the density function $\Pi(\rho)$ that was in (7)-(9), (15):

$$U = \rho \mathbf{g}\mathbf{x} + \Pi(\rho) - \Pi(\rho_0),$$

where $\rho_0(\mathbf{x})$ is the equilibrium density. The evolution equations for the generalized Clebsch potentials (15) and (22) can be written in the canonical Hamilton form if (23) is selected as the Hamiltonian (see [2, 10]):

$$\frac{\partial \lambda}{\partial t} = \frac{\delta H}{\delta \rho}, \ \frac{\partial \rho}{\partial t} = -\frac{\delta H}{\delta \lambda}, \ \frac{\partial \chi}{\partial t} = \frac{\delta H}{\delta \psi}, \ \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \chi}. \tag{24}$$

The Hamiltonian form of the general equation for the Lamb momentum density is obtained below in Euler and Lagrange variables. Noncanonical Poisson parentheses [14] are used in the Euler variables. For our case it is simplest to obtain their form by the method of conversion from the canonical variables [10]. In conformity with (24), the Poisson parentheses of two arbitrary functionals F and G of the canonical variables have the form

$$\{F, G\}(\lambda, \rho, \chi, \psi) = \int \left(\frac{\delta F}{\delta \lambda}\frac{\delta G}{\delta \rho} - \frac{\delta F}{\delta \rho}\frac{\delta G}{\delta \lambda} + \frac{\delta F}{\delta \chi}\frac{\delta G}{\delta \psi} - \frac{\delta F}{\delta \psi}\frac{\delta G}{\delta \chi} \right) dV. \tag{25}$$

We go over to the variables ρ, $\mathbf{q} = \lambda \nabla \rho + \chi \nabla \psi$:

$$\frac{\delta F}{\delta \rho} = \frac{\delta F}{\delta \rho}\Big|_{\mathbf{q}} - \operatorname{div}\left(\lambda \frac{\delta F}{\delta \mathbf{q}}\right), \ \frac{\delta F}{\delta \lambda} = \frac{\delta F}{\delta \mathbf{q}} \nabla \rho, \tag{26}$$
$$\frac{\delta F}{\delta \lambda} = \frac{\delta F}{\delta \mathbf{q}} \nabla \psi, \quad \frac{\delta F}{\delta \psi} = -\operatorname{div}\left(\chi \frac{\delta F}{\delta \mathbf{q}}\right).$$

Substituting (26) into (24), we find the Poisson parentheses in terms of the fields ρ, $\mathbf{q}$

$$\{F, G\}(\rho, \mathbf{q}) = -\int q_j \left(\frac{\delta F}{\delta q_m} \nabla_m \frac{\delta G}{\delta q_j} - \frac{\delta G}{\delta q_m} \nabla_m \frac{\delta F}{\delta q_j} \right) dV + \int \frac{\partial \rho}{\partial x_j} \left(\frac{\delta F}{\delta q_j}\frac{\delta G}{\delta \rho} - \frac{\delta F}{\delta \rho}\frac{\delta G}{\delta q_j} \right) dV. \tag{27}$$

By construction, (27) satisfies all the requirements imposed on Poisson parentheses. The expression (27) can be taken as the definition of the Poisson parentheses in the general case in which $\mathbf{q}$ is not expressed in terms of canonical variables while the spirality of the field $\mathbf{q}'$ takes on arbitrary values. Utilizing (27), we find that the equation $d\rho/dt = 0$ and Eq. (10) can be written in the noncanonical Hamiltonian form: $\partial \rho/\partial t = \{\rho, H\}$, $\partial \mathbf{q}/\partial t = \{\mathbf{q}, H\}$. As H is varied, it is understood that Φ is a functional of ρ, $\mathbf{q}$ such that $\operatorname{div} \mathbf{v} = 0$.

In the particular case of $\rho = \text{const}$ the potential energy in the Hamiltonian (23) must be discarded and the second term in the Poisson parentheses as well. In this case the Hamiltonian can be written by using the projection operator (12):

$$H = (1/2\rho) \int Q_{ij}(\mathbf{x}', \mathbf{x}'')\, q_i(\mathbf{x}')\, q_j(\mathbf{x}'')\, dV(\mathbf{x}')\, dV(\mathbf{x}''), \tag{28}$$

which permits taking automatic account of the condition $\operatorname{div} \mathbf{v} = 0$.

The motion equation can be written in Hamiltonian form even in the Lagrange variables. Expressing the Euler coordinates x in terms of the Lagrangian **a** in (28), we obtain

$$H = (1/2\rho) \int Q_{ij}[\mathbf{x}(\mathbf{a}'), \mathbf{x}(\mathbf{a}'')]\, \tilde{q}_i(\mathbf{a}')\, \tilde{q}_j(\mathbf{a}'')\, dV(\mathbf{a}')\, dV(\mathbf{a}''), \tag{29}$$
$$\tilde{q}_i(\mathbf{a}) = q_i[\mathbf{x}(\mathbf{a})].$$

We calculate the functional derivatives of the Hamiltonian (29) with respect to $x_i(\mathbf{a})$ and $\tilde{q}_i(\mathbf{a})$:

$$\frac{\delta H}{\delta x_i(\mathbf{a})} = \tilde{q}_j \frac{\partial}{\partial x_i} \int Q_{jm}[\mathbf{x}(\mathbf{a}), \mathbf{x}(\mathbf{a}')]\, \tilde{q}_m(\mathbf{a}')\, dV(\mathbf{a}'),$$
$$\frac{\delta H}{\delta \tilde{q}_i(\mathbf{a})} = \int Q_{im}[\mathbf{x}(\mathbf{a}), \mathbf{x}(\mathbf{a}')]\, \tilde{q}_m(\mathbf{a}')\, dV(\mathbf{a}').$$

It hence follows that the motion equations of an inviscid fluid of constant density (10) can be written in canonical Hamiltonian form in Lagrange variables:

$$\partial x_i/\partial t = \delta H/\delta \tilde{q}_i, \ \partial \tilde{q}_i/\partial t = -\delta H/\delta x_i. \tag{30}$$

In the case $\rho \neq$ const the density in Lagrange variables does not change in time. Consequently, another calibration of the Lamb momentum density is convenient in Lagrange coordinates, in which the motion equations do not contain Euler density gradients. If we require that the field in (7) satisfy the condition $d\varphi/dt = p$ then the equation for $\mathbf{q}$ acquires the form

$$dq_i/dt = -q_j \partial v_j/\partial x_i + \rho \partial (v^2/2)/\partial x_i - \rho g_i.$$

The right side of this equation equals $-\delta H/\delta \mathbf{x}$. The functional derivative of the Hamiltonian (23) with respect to $\tilde{q}_i$ yields the velocity. Therefore, (30) remains valid even in the case $\rho \neq$ const.

CONCLUSION

Let us discuss the connection between the representation (16) and the Weber transformation of the hydrodynamics equations [11]. As is known, the equation for the vorticity in an ideal homogeneous fluid can be rewritten in the form of the Cauchy equations [11]:

$$\omega_i(t) = \omega_j(0) \partial x_i/\partial a_j. \tag{31}$$

In an inhomogeneous fluid (13) can be written in the form (31). In some sense, Eq. (17) is the conjugate to Eq. (13). Hence, it should be expected that $\mathbf{q}'$ will evolve according to an equation conjugate to (31).

Let us make the following change of variable in (17)

$$q_i' = b_j \partial a_j/\partial x_i, \tag{32}$$

where $\mathbf{b}$ is a new unknown vector. Substituting (32) into (17) yields $d\mathbf{b}/dt = 0$. Hence, by setting $t = 0$ in (32), we obtain $q'(0) = \mathbf{b}(0) = \mathbf{b}(t)$ and

$$q_i'(t) = q_j'(0)\, \partial a_j/\partial x_i. \tag{33}$$

The equality

$$\rho v_j \partial x_j/\partial a_i = -\partial \varphi/\partial a_i + \lambda \partial \rho/\partial a_i + q_i'(0)$$

follows from (16) and (33) and can be considered as one of the forms of the generalized Weber transformations.

Going over from the field $\mathbf{v}$ to the new variable $\mathbf{q}$ can be considered as the replacement of the hydrodynamic field calibration in which a new condition (8) is imposed instead of the incompressibility calibration condition. An analogous change in calibration is possible in magnetohydrodynamics equations (see [4]) as well as in compressible fluid hydrodynamics with the appropriate generalization of the scheme to search for Lagrange invariants. Lagrange invariants similar to those obtained above are missing in the presence of viscosity. In this case the equation for the Lamb momentum density is derived analogously to (9) and (10). The dissipative component $\nu \Delta \mathbf{v}$ will be in their right sides. In the case of a homogeneous fluid $\nu \Delta \mathbf{v} = -\nu$ curl curl $\mathbf{v} = -\nu$ curl curl $\mathbf{q}/\rho$.

Proposals to use a finite-dimensional approximation of the flow by vortex rings are contained in [15-17]. To produce gridless algorithms for the computation it was here proposed in [16, 17] to use a system of dynamical equations for the Lamb momentums and the coordinates of small vortex rings which had been derived for the homogeneous fluid case in [16, 18]. Necessary for the applicability of this system is that the dimensionless parameter, the ratio of the vortex dimensions to the spacing between them, be small. Consequently, it does not describe the vortex trajectories at intervening spacings on the order of their size. There is also a considerable arbitrariness in the selection of the vortex ring parameters, and an associated arbitrariness in the magnitude of their self-induced velocity.

The dynamical equation (9) can be used to construct analogous numerical algorithms for the computation of ideal fluid flows free of the constraints noted as well as to extend them to the case of an inhomogeneous fluid. That (9) is Hamiltonian in form means that the phase volume and other Poincaré integral invariants are conserved in time. This property permits the construction of a fluid statistical mechanics and can be useful in investigating the stability of solutions.

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